

ALGEBRAIC NUMBER THEORY-SUMMER SCHOOL NOTES

Contents

1. Lecture 1: July 30: Introduction	1
1.1. Diophantine equation	1
1.2. Algebraic Integers	3
1.3. Discriminant and integral basis	3
1.4. Exercise Sheet 1.	5
2. Lecture 2: July 31	6
2.1. Cyclotomic fields	6
2.2. Exercise sheet 2.	7
3. Lecture 3: August 2	8
3.1. Dedekind domain	8
3.2. Fractional ideal.	8
3.3. Localization	9
3.4. Extension of Dedekind domains.	9
3.5. Exercise sheet 3.	10
4. Lecture 4: August 5	11
4.1. Galois extension	11
4.2. Chebotarev density theorem	12
4.3. Applications to cyclotomic fields.	12
4.4. Exercise sheet 4.	13
5. Lecture 5: August 6	14
5.1. Exercise sheet 5.	17
6. Lecture 6: August 9	18
6.1. Variation of the class numbers in families	18
6.2. Dirichlet's unit theorem	19
6.3. Exercise sheet 6.	21
References	21

1. Lecture 1: July 30: Introduction

1.1. Diophantine equation.

Example 1.1. Consider the equation

$$x^2 + y^2 = 1$$

in the rational number field \mathbb{Q} .

We know that (1;00509 0 Td [(:)]TJ/F88

1.2. Algebraic Integers.

Definition 1.9. Let $A \subseteq B$ be an extension of rings. $x \in B$ is called *integral* over A if x satisfies

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

with $a_i \in A$. B is *integral* over A if all $x \in B$

Here non-degenerate means that if $\text{Tr}_{L=K}(xy) = 0$ for all $y \in L$, then $x = 0$: Or equivalently to say, $L \subset \text{Hom}_K(L; K)$ induced by $\text{Tr}_{L=K}$.

1.4. Exercise Sheet 1.

Exercise 1.

3. Lecture 3: August 2

3.1. **Dedekind domain.** A is called a Dedekind domain if A is Noetherian, integrally closed and each nonzero prime ideal is maximal.

Note that localization of a Dedekind domain is also Dedekind. We always have the following setting in this section. Let A be a Dedekind domain, let K be the fraction field of A . Let $L=K$ be a finite separable extension. Let B be the integral closure of A in L .

Proposition 3.1. *Let A be a Dedekind domain, let K be the fraction field of A . Let $L=K$ be a finite separable extension. Let B be the integral closure of A in L . Then B is a Dedekind domain.*

Proof. Of course B is integrally closed. Let $\mathfrak{p}A$

Remark 3.8. There are Dedekind domains A such that Cl_A is infinite.

Example 3.9. $A = \mathbb{C}[x, y]/(y^2)$

(1) p ramified in K , i.e. some $e_i > 1$, i

$$p \nmid j \quad \kappa = 4d - d - 2;$$

4. Lecture 4: August 5

Let $L=K$ be a finite extension of number fields with integer ring B and A , and $\mathfrak{p} \in 0$ be a prime of O_K . We have a prime decomposition

\mathfrak{p}

4.4. Exercise sheet 4.

Exercise 1. Use the cyclotomic extension $\mathbb{Q}(\zeta_8)$ to show the quadratic reciprocity law for 2: if p is an

5. Lecture 5: August 6

Let K be a number field and

$$\text{Cl}_K = I_{O_K} / P_{O_K}$$

where I_{O_K} is the free group generated by fractional ideals of O_K with P_{O_K} the subgroup consisting of principal fractional ideals.

Theorem 5.1.

Note that for any number field K , the Minkowski constant C_K

Theorem 5.11. *For a number field K , there exists a unique finite extension*

6. Lecture 6: August 9

6.1.

Theorem 6.11 (Iwasawa). *Let $K_1 = K$ be a Z_p*

Let $\phi: K \rightarrow R^{r_1} \times C^{r_2} =: V$, V is a R -algebra, and $V = (R)^{r_1} \times (C)^{r_2}$. For any element $(y_i; z_j) \in V$, define norm to be $N(y_i; z_j) = \prod_i y_i \prod_j z_j^2$. Then for any $\alpha \in O_K$, $N(\alpha) \in \mathbb{Z}$.

6.3. Exercise sheet 6.

Exercise 1. *The aim of this exercise is to give a proof of the finiteness of the ideal class group of a number field K*